Transformation Laws for Theta functions

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1 Introduction

We prove some results which extend the classical theory, due to Hecke-Schoeneberg [H], [S1], of the transformation laws of theta functions. Although our results are classical in natural, they were suggested by recent work involving modular-invariance in conformal field theory [DMN], and we shall say more about these connection in the last section of the present paper.

Let Q be a positive-definite, integral quadratic form of even rank $f = 2r$ with theta-function

$$\theta(Q, \tau) = \sum_{m \in \mathbb{Z}^f} e^{2\pi i Q(m)\tau}. \tag{1.1}$$

One knows [S2] that $\theta(Q, \tau)$ is a modular form of weight r and character ϵ on the group $\Gamma_0(N)$. Here N is the level of Q and ϵ is the Dirichlet character given by the Jacobi symbol

$$\epsilon(n) = \left(\frac{(-1)^r \det A}{n} \right)$$

for $n > 0$, with A a Gram matrix of the bilinear form $\langle , \rangle$ corresponding to Q.

We fix a vector $v \in \mathbb{C}^f$ and define

$$\theta(Q, v, k, \tau) = \sum_{m \in \mathbb{Z}^l} \langle v, m \rangle^k e^{2\pi i Q(m)\tau} \tag{1.2}$$

where k is a nonnegative integer. Obviously, $\theta(Q, v, k, \tau)$ identically zero if k is odd, and coincides with $\theta(Q, \tau)$ if $k = 0$.

We define

$$\Theta(Q, v, \tau, X) = \sum_{n \geq 0} \frac{2^n \theta(Q, v, 2n, \tau)}{(2n)!} (2\pi i X)^n \tag{1.3}$$

regarding this as a function on $\mathfrak{h} \times \mathbb{C}$ ($\mathfrak{h}$ is the complex upper half-plane) for fixed Q and v.

Our main result may then be stated as follows:

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Theorem 1 *$\Theta(Q,v,\tau,X)$ satisfies the following transformation law for $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ in $\Gamma_0(N)$:*

$$\Theta(Q,v,\gamma\tau,\frac{X}{(c\tau+d)^2}) = \epsilon(d)(c\tau+d)^r \exp\left(\frac{c\langle v,v\rangle X}{c\tau+d}\right)\Theta(Q,v,\tau,X). \tag{1.4}$$

It is evident from (1.2) that scaling v (i.e., replacing v by λv for a scalar λ) simply multiplies $\theta(Q,v,k,\tau)$ by λ^k. For this reason, there are essentially only two cases, namely

(a) v is a null vector i.e., $\langle v,v\rangle = 0$, or

(b) v is a unit vector i.e., $\langle v,v\rangle = 1$.

Suppose first that v is a null vector. Then Hecke [H] proved that the function $P_k(x) = \langle v,x\rangle^k$ is a spherical harmonic of degree k with respect to Q, moreover every spherical harmonic of degree k is a linear combination of such functions. Thus in this case (1.2) is simply a theta function with spherical harmonic $\theta(Q,P_k,\tau)$, and

$$\Theta(Q,v,\tau,X) = \sum_{n\geq 0}\frac{2^n\theta(Q,P_{2n},\tau)}{(2n)!}(2\pi iX)^n. \tag{1.5}$$

The transformation law (1.4) then says exactly that $\theta(Q,P_{2n},\tau)$ is a modular form on $\Gamma_0(N)$ of weight $r+2n$ and character ϵ. This is the theorem of Schoeneberg [S1].

Suppose next that v is a unit vector. Then the transformation law (1.4) says that $\Theta(Q,v,\tau,X)$ is a (holomorphic) Jacobi-like form of weight r, level N, character ϵ in the sense of Zagier [Z].

When v is a unit vector, $\theta(Q,v,k,\tau)$ will not be modular, but one can suitably combine two Jacobi-like forms to produce a sequence of modular forms (loc. cit.). For example, there is a holomorphic Jacobi-like form of weight 0 of particular interest, namely

$$\tilde{E}_2(\tau,X) = \sum_{n\geq 0}(-1)^n\frac{E_2(\tau)^n}{n!}(2\pi iX)^n \tag{1.6}$$

where

$$E_2(\tau) = -\frac{1}{12} + 2\sum_{n\geq 1}\sigma_1(n)q^n$$

is the usual "unmodular" Eisenstein series of "weight" 2 normalized as indicated. Using this together with Theorem 1 yields

Theorem 2 *Let notation be as before, and suppose that v is a unit vector. Set*

$$\gamma(t,k) = 2^{-t}\binom{k}{t}\binom{k-t}{t}t! \tag{1.7}$$

$$\Psi(Q,v,2k,\tau)=\sum_{t=0}^{k}\gamma(t,2k)E_2(\tau)^t\theta(Q,v,2k-2t,\tau). \tag{1.8}$$

Then $\Psi(Q,v,2k,\tau)$ is a holomorphic modular form on $\Gamma_0(N)$ of weight $2k+r$ and character ϵ.

One knows that if $k>0$ and v is a null vector then in fact $\theta(Q,P_k,\tau)$ is a cusp form. In the same spirit we have the following supplement to Theorem 2:

Theorem 3 *Let notation and assumptions be as in Theorem 2, and $k\geq 2$. Then*

$$\Psi(Q,v,2k,\tau)-\gamma(k,2k)(-\frac{1}{12})^k\theta(Q,\tau)E_{2k}(\tau) \tag{1.9}$$

is a cusp form on $\Gamma_0(N)$ of weight $2k+r$ and character ϵ. Here,

$$E_{2k}(\tau)=1-\frac{4k}{B_{2k}}\sum_{n\geq 1}\sigma_{k-1}(n)q^n$$

is the usual Eisenstein series.

It is forms of the general shape (1.9) that appear as partition function in certain conformal field theories [DMN], and whose existence led us to the results of the present paper.

The paper is organized as follows: in Section 2 we discuss Jacobi-like forms and the proof of Theorem 2. Section 3 is devoted to the proof of Theorems 1 and 3, which follows in general outline the original proof of Schoeneberg. In Section 4 we discuss connections with conformal field theory and state some further results which will be proved in [DMN].

2 Jacobi-like forms

We are interested in holomorphic functions $\phi(\tau,X)$ on $\mathfrak{h}\times\mathbb{C}$ of the form

$$\phi(\tau,X)=\sum_{n\geq 0}\phi^{(n)}(\tau)(2\pi iX)^n \tag{2.1}$$

and which satisfy

$$\phi(\gamma\tau,\frac{X}{(c\tau+d)^2})=\chi(d)(c\tau+d)^k\exp\left(\frac{cmX}{c\tau+d}\right)\phi(\tau,X) \tag{2.2}$$

for some $m\in\mathbb{C}$, integer k and Dirichlet character $\chi(\mathrm{mod}\,N)$, and for all $\gamma\in\begin{pmatrix}a & b\\ c & d\end{pmatrix}\in$ $\Gamma_0(N)$. By scaling X, the essential cases correspond to $m=0$ and $m=1$. As long as

ϕ is holomorphic at the cusps, the case $m = 0$ means precisely that each $\phi^{(n)}(\tau)$ is a holomorphic modular form on $\Gamma_0(N)$ of weight $k + 2n$ and character χ. The case $m = 1$ means that ϕ is a holomorphic Jacobi-like form on $\Gamma_0(N)$ of weight k and character χ (cf. [Z]).

By way of examples, let $\tilde{E}_2(\tau, X)$ be as in (1.6). The particular normalization of $E_2(\tau)$ that we are using satisfies the well-known transformation law

$$E_2(\gamma\tau) = (c\tau + d)^2 E_2(\tau) - \frac{c(c\tau + d)}{2\pi i}. \tag{2.3}$$

It follows that $\tilde{E}_2(\tau, X) = \exp(-2\pi i E_2(\tau)X)$ satisfies

$$\begin{aligned}\tilde{E}_2(\gamma\tau, \frac{X}{(c\tau + d)^2}) &= \exp\left(-2\pi i \left((c\tau + d)^2 E_2(\tau) - \frac{c(c\tau + d)}{2\pi i}\right) \frac{X}{(c\tau + d)^2}\right) \\ &= \exp\left(\frac{cX}{c\tau + d}\right) \tilde{E}_2(\tau, X),\end{aligned}$$

so that $\tilde{E}_2(\tau, X)$ is a holomorphic Jacobi-like form of level 1 and weight 0.

Now let $\Theta(Q, v, \tau, X)$ be as in (1.3) with v a unit vector v. Then $\tilde{E}_2(\tau, -X)\Theta(Q, v, \tau, X)$ satisfies (2.2) with $m = 0$ (we are assuming the truth of Theorem 1 at this point). We have

$$\tilde{E}_2(\tau, -X)\Theta(Q, v, \tau, X) = \sum_{k \geq 0} f^{(k)}(\tau)(2\pi i X)^k$$

where

$$f^{(k)}(\tau) = \frac{2^k}{(2k)!} \sum_{t=0}^{k} \gamma(t, 2k) E_2(\tau)^t \theta(Q, v, 2k - 2t, \tau) \tag{2.4}$$

and $\gamma(t, 2k)$ is as in (1.7). It follows, granting holomorphy at the cusps for now, that $f^{(k)}(\tau)$ is a holomorphic modular form on $\Gamma_0(N)$ of weight $r + 2k$ and character ϵ. So Theorem 2 follows from Theorem 1.

There are other Jacobi-like forms that one could use in place of $\tilde{E}_2(\tau, X)$ in order to construct modular forms involving the $\theta(Q, v, n, \tau)$. For example, we could take the Cohen-Kuznetsov Jacobi-like form

$$\sum_{n \geq 0} \frac{f^{(n)}(\tau)}{n!(n + k - 1)!} (2\pi i X)^n$$

for f a modular form of weight k, as described in [Z]. We will not pursue this possibility here: it is the forms $\Psi(Q, v, 2k, \tau)$ of Theorem 2 that we need in [DMN].

3 Proofs of Theorem 1 and 3

In this section we present the proofs of Theorem 1 and 3. They follow in general outline the proof of Schoeneberg [S2]. We therefore adopt notation similar to (loc. cit.) and omit some details. In particular, we have $\langle x, y\rangle = x'Ay$ for $x, y \in \mathbb{C}^f$, where x' denotes the transpose of the column vector x.

Let A and Q be as in Section 1. For $x = (x_1, ..., x_f)$ and a scalar λ we set

$$\theta_\lambda(A, x) = \sum_{m\in\mathbb{Z}^f} \exp(2\lambda Q(m + x)). \tag{3.1}$$

For $l = (l_1, ..., l_f) \in \mathbb{C}^f$ we let $\mathcal{L}$ be the linear differential operator

$$\mathcal{L} = \sum_{i=1}^{f} l_i \frac{\partial}{\partial x_i}. \tag{3.2}$$

Lemma 3.1 *Let $k \geq 0$ be an integer. Then*

$$\mathcal{L}^k(\theta_\lambda(A, x)) = \sum_{i=0}^{[k/2]} \sum_{m\in\mathbb{Z}^f} \gamma(i, k)(2\lambda)^{k-i}(2Q(l))^i(l'A(m + x))^{k-2i} \exp(2\lambda Q(m + x)) \tag{3.3}$$

where $\gamma(i, k)$ is defined by (1.7).

Proof: One sees that there is an equality of the form

$$\mathcal{L}^k(\theta_\lambda(A, x)) = \sum_{i=0}^{[k/2]} \sum_{m\in\mathbb{Z}^f} \gamma_\lambda(i, k)(2Q(l))^i(l'A(m + x))^{k-2i} \exp(2\lambda Q(m + x)) \tag{3.4}$$

for some scalars $\gamma_\lambda(i, k), 0 \leq i \leq [k/2], k \geq 0$. Setting $\gamma_\lambda(i, k) = 0$ for values of i and k not in these ranges, $\gamma_\lambda(i, k)$ satisfies a recursion relation, namely

$$\gamma_\lambda(i, k + 1) = (k + 2 - 2i)\gamma_\lambda(i - 1, k) + 2\lambda\gamma_\lambda(i, k), \gamma_\lambda(0, 0) = 1. \tag{3.5}$$

We can solve the recursion, and find that $\gamma_\lambda(i, k) = (2\lambda)^{k-i}\gamma(i, k)$. The lemma follows. $\square$

Now one knows (e.g. page 205 of [S2]) that the following transformation law holds: for τ in the upper half-plane and for a suitable determination of the square root,

$$\sum_{m\in\mathbb{Z}^f} \exp(2\pi i\tau Q(m + x)) = \theta_{\pi i\tau}(A, x)$$
$$= \frac{1}{(\sqrt{-i\tau})^f(\det A)^{1/2}} \sum_{m\in\mathbb{Z}^f} \exp(-\frac{\pi i}{\tau}m'A^{-1}m + 2\pi i m'x). \tag{3.6}$$

We apply the operator $\mathcal{L}^k$ to both sides of (3.6), using Lemma 3.1, to obtain

$$\sum_{j=0}^{[k/2]} \sum_{m\in\mathbb{Z}^f} (2\pi i\tau)^{k-j}\gamma(j,k)(2Q(l))^j(l'A(m+x))^{k-2j}\exp(2\pi i\tau Q(m+x))$$
$$= \frac{1}{(\sqrt{-i\tau})^f(\det A)^{1/2}} \sum_{m\in\mathbb{Z}^f}(2\pi i m'l)^k\exp(-\frac{\pi i}{\tau}m'A^{-1}m+2\pi i m'x). \tag{3.7}$$

Recall that N is the level of A. Following [S2], we replace x by h/N, τ by $-1/\tau$ and m by Am_1/N on the r.h.s. of (3.7). Remembering that $f=2r$ and setting $D=\det A$, we get

$$\frac{(2\pi i)^k\tau^r N^{-k}}{i^r\sqrt{D}} \sum_{\substack{m_1\in\mathbb{Z}^f\\ Am_1=0(N)}} (l'Am_1)^k\exp(2\pi i\tau Q(m_1)/N^2+2\pi i m_1 Ah/N^2)$$
$$= \sum_{j=0}^{[k/2]} \sum_{\substack{m\in\mathbb{Z}^f\\ m\equiv h(N)}} N^{2j}\left(\frac{-2\pi i}{\tau}\right)^{k-j}\gamma(j,k)(2Q(l))^j(l'Am)^{k-2j}\exp(\frac{-2\pi i}{\tau}\frac{Q(m)}{N^2}). \tag{3.8}$$

Note that if l is a null vector, only the term with $j=0$ survives on the r.h.s. of (3.8), which then reduces to equation (12) of [S2], page 209.

We discuss some transformation formulas. For $h\in\mathbb{Z}^f$, $Ah\equiv 0 \pmod N$ and $0\le j\le k$, we set

$$\theta(A,h,l,k,\tau) = \frac{1}{N^k}\sum_{\substack{m\in\mathbb{Z}^f\\ m\equiv h(N)}} (l'Am)^k\exp(2\pi i\tau Q(m)/N^2). \tag{3.9}$$

It is also convenient to introduce

$$\Theta(A,h,l,k,j,\tau) = \frac{(-i)^{r+2k}\tau^{r+k-2j}}{\sqrt{D}} \sum_{\substack{g \bmod N\\ Ag\equiv 0(N)}} \exp(2\pi i g'Ah/N^2)\theta(A,g,l,k-2j,\tau). \tag{3.10}$$

Remark 3.2 (i) It is clear that $\Theta(A,h,l,k,j,\tau)$ depends only on $k-2j$, rather than both k and j. However it is convenient to keep the notation as it is.

(ii) Note that if $h=0$ then $\theta(A,h,l,k,\tau)$ is just the function $\theta(A,l,k,\tau)=\theta(Q,l,k,\tau)$.

Theorem 3.3 *We have*

$$\theta(A,h,l,k,-1/\tau) = \sum_{j=0}^{[k/2]}\left(\frac{Q(l)\tau}{\pi i}\right)^j\gamma(j,k)\Theta(A,h,l,k,j,\tau). \tag{3.11}$$

Proof: As $Ah \equiv 0 \bmod N$, we can split-off an exponential factor form the l.h.s. of (3.8). Then using (3.9), (3.8) can be written in the form

$$\frac{(2\pi i)^k \tau^r}{i^r \sqrt{D}} \sum_{\substack{g \bmod N \\ Ag \equiv 0(N)}} \exp(2\pi i g' Ah/N^2)\theta(A,g,l,k,\tau)$$

$$= \sum_{j=0}^{[k/2]} \left(\frac{-2\pi i}{\tau}\right)^{k-j} \gamma(j,k)(2Q(l))^j \theta(A,h,l,k-2j,-1/\tau). \tag{3.12}$$

We shall prove Theorem 3.3 by induction on k. If $k=0$ it reduces to a standard transformation law (equation 17II of [S2], page 210). In the general case, (3.12) yields

$$\theta(A,h,l,k,-1/\tau) = \frac{(-i)^{r+2k}\tau^{r+k}}{\sqrt{D}} \sum_{\substack{g \bmod N \\ Ag \equiv 0(N)}} \exp(2\pi i g' Ah/N^2)\theta(A,g,l,k,\tau)$$

$$- \sum_{j=1}^{[k/2]} \gamma(j,k) \left(-\frac{Q(l)\tau}{\pi i}\right)^j \theta(A,h,l,k-2j,-1/\tau)$$

$$= \Theta(A,h,l,k,0,\tau)$$

$$- \sum_{j=1}^{[k/2]} \gamma(j,k) \left(\frac{-Q(l)\tau}{\pi i}\right)^j \sum_{t=0}^{[(k-2j)/2]} \gamma(t,k-2j) \left(\frac{Q(l)\tau}{\pi i}\right)^t \Theta(A,h,l,k-2j,t,\tau).$$

As $\Theta(A,h,l,k-2j,t,\tau) = \Theta(A,h,l,k,t+j,\tau)$, we see that $\theta(A,h,l,k,-1/\tau)$ is equal to

$$\Theta(A,h,l,k,0,\tau) + \sum_{u=1}^{[k/2]} \left(\frac{Q(l)\tau}{\pi i}\right)^u \beta(u,k)\Theta(A,h,l,k,u,\tau) \tag{3.13}$$

where

$$\beta(u,k) = -\sum_{j=1}^{[k/2]} (-1)^j \gamma(j,k)\gamma(u-j,k-2j). \tag{3.14}$$

From (1.7) and (3.14) we see that

$$\beta(u,k) = \gamma(u,k) - \sum_{j=0}^{[k/2]} (-1)^j \gamma(j,k)\gamma(u-j,k-2j) = \gamma(u,k) - \sum_{j=0}^{[k/2]} (-1)^j \gamma(u,k) \binom{u}{j}$$

i.e., $\beta(u,k) = \gamma(u,k)$. Now (3.13) implies the desired equality (3.11). $\square$

We now proceed to our main transformation formula, which is the following:

Theorem 3.4 *If* $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ *lies in* $\Gamma_0(N)$, *and if* $d>0$, *then*

$$(c\tau+d)^{-(r+k)}\theta(A,h,l,k,\frac{a\tau+b}{c\tau+d})$$

$$= \exp(2\pi i Q(h)ab/N^2)\epsilon(d) \sum_{j=0}^{[k/2]} \left(\frac{Q(l)c}{\pi i(c\tau+d)}\right)^j \gamma(j,k)\theta(A,bh,l,k-2j,\tau). \tag{3.15}$$

In particular, taking $h=0$, if $d>0$ then

$$(c\tau+d)^{-(r+k)}\theta(A,l,k,\frac{a\tau+b}{c\tau+d}) = \epsilon(d)\sum_{j=0}^{[k/2]}\left(\frac{Q(l)c}{\pi i(c\tau+d)}\right)^j \gamma(j,k)\theta(A,l,k-2j,\tau). \quad (3.16)$$

We begin by noting that

$$\theta(A,h,l,k,\tau+1) = \exp(2\pi i Q(h)/N^2)\theta(A,h,l,k,\tau), \quad (3.17)$$

and also if $c>0$ then

$$\theta(A,h,l,k,\tau) = \sum_{\substack{g \bmod cN \\ g\equiv h(N)}} \theta(cA,g,l,k,c\tau). \quad (3.18)$$

(3.17) is immediate from (3.9) (remembering that $Ah=0 \pmod N$); (3.18) follows as in equation 18 of [S2], page 211.

Using (3.17), (3.18) and Theorem 3.3 we calculate for $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ in $SL(2,\mathbb{Z})$ with $c>0$ that

$$\begin{aligned}
&\theta(A,h,l,k,\gamma\tau) = \theta(A,h,l,k,c^{-1}(a-(c\tau+d)^{-1})) \\
&= \sum_{\substack{g \bmod cN \\ g\equiv h(N)}} \theta(cA,g,l,k,a-(c\tau+d)^{-1}) \\
&= \sum_{\substack{g \bmod cN \\ g\equiv h(N)}} \exp(2\pi i acQ(g)/c^2N^2)\theta(cA,g,l,k,-(c\tau+d)^{-1}) \\
&= \sum_{\substack{g \bmod cN \\ g\equiv h(N)}} \sum_{j=0}^{[k/2]} \exp(2\pi i aQ(g)/cN^2)\left(\frac{Q(l)c(c\tau+d)}{\pi i}\right)^j \gamma(j,k)\Theta(cA,g,l,k,j,c\tau+d) \\
&= \sum_{\substack{g \bmod cN \\ g\equiv h(N)}} \sum_{j=0}^{[k/2]} \sum_{\substack{q \bmod cN \\ cAq\equiv 0(cN)}} \exp(2\pi i aQ(g)/cN^2)\left(\frac{Q(l)c(c\tau+d)}{\pi i}\right)^j \gamma(j,k)\frac{(-i)^{r+2(k-2j)}}{\sqrt{c^f D}} \cdot \\
&\qquad \cdot (c\tau+d)^{r+k-2j}\exp(2\pi i g'Aq/cN^2)\theta(cA,q,l,k-2j,c\tau+d) \\
&= \frac{(c\tau+d)^{r+k}(-i)^{r+2k}}{c^r\sqrt{D}} \sum_{\substack{g \bmod cN \\ g\equiv h(N)}} \sum_{\substack{q \bmod cN \\ Aq\equiv 0(N)}} \sum_{j=0}^{[k/2]} \exp(2\pi i(aQ(g)+dQ(q)+g'Aq)/cN^2) \cdot \\
&\qquad \cdot\gamma(j,k)\left(\frac{cQ(l)}{(c\tau+d)\pi i}\right)^j \theta(cA,q,l,k-2j,c\tau) \qquad (3.19)
\end{aligned}$$

where we have used the fact that cA has level cN.

Following [S2], page 213 we write

$$\phi_{h,q} = \sum_{\substack{g \bmod cN \\ g\equiv h(N)}} \exp(2\pi i(aQ(g) + dQ(q) + g'Aq)/cN^2) \tag{3.20}$$

and note that $\phi_{h,q}$ depends on q only modulo N. Then (3.19) can be put into the form

$$\frac{(c\tau+d)^{r+k}(-i)^{r+2k}}{c^r\sqrt{D}} \sum_{j=0}^{[k/2]} \sum_{\substack{q \bmod cN \\ Aq\equiv 0(N)}} \phi_{h,q}\gamma(j,k)\left(\frac{cQ(l)}{(c\tau+d)\pi i}\right)^j \theta(cA,q,l,k-2j,c\tau)$$

$$= \frac{(c\tau+d)^{r+k}(-i)^{r+2k}}{c^r\sqrt{D}} \sum_{j=0}^{[k/2]} \sum_{q_1\, n\bmod N} \phi_{h,q_1}\gamma(j,k)\left(\frac{cQ(l)}{(c\tau+d)\pi i}\right)^j \cdot$$
$$\cdot \sum_{\substack{q \bmod cN \\ Aq\equiv 0(N), q\equiv q_1(N)}} \theta(cA,q,l,k-2j,c\tau).$$

Another application of (3.18) now yields

Lemma 3.5 *Let* $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{Z})$ *with* $c>0$. *Then*

$$(c\tau+d)^{-(r+k)}\theta(A,h,l,k,\gamma\tau)$$
$$= \frac{(-i)^{r+2k}}{c^r\sqrt{D}} \sum_{j=0}^{[k/2]} \sum_{\substack{q_1 \bmod N \\ Aq_1\equiv 0(N)}} \phi_{h,q_1}\gamma(j,k)\left(\frac{cQ(l)}{(c\tau+d)\pi i}\right)^j \theta(A,q_1,l,k-2j,\tau). \tag{3.21}$$

We continue the calculation, but now assuming also that $d\equiv 0 \pmod N$. As in (loc. cit.) we now find that (3.21) can be written as follows:

$$(c\tau+d)^{-(r+k)}\theta(A,h,l,k,\gamma\tau) = \frac{(-i)^{r+2k}\phi_{h,0}}{c^r\sqrt{D}} \sum_{j=0}^{[k/2]} \sum_{\substack{q_1 \bmod N \\ Aq_1\equiv 0(N)}} \exp(-2\pi i h' Aq_1 b/N^2)\cdot$$
$$\cdot\gamma(j,k)\left(\frac{cQ(l)}{(c\tau+d)\pi i}\right)^j \theta(A,q_1,l,k-2j,\tau). \tag{3.22}$$

Replacing τ by $-1/\tau$ in (3.22) and using Theorem 3.3 again leads to

$$\left(\frac{d\tau-c}{\tau}\right)^{-(r+k)} \theta(A,h,l,k,\frac{b\tau-a}{d\tau-c})$$
$$= \frac{(-i)^{r+2k}\phi_{h,0}}{c^r\sqrt{D}} \sum_{j=0}^{[k/2]} \sum_{\substack{q_1 \bmod N \\ Aq_1\equiv 0(N)}} \sum_{u=0}^{[(k-2j)/2]} \exp(-2\pi i h' Aq_1 b/N^2)\gamma(j,k)\cdot$$

$$\cdot \left(\frac{cQ(l)\tau}{(d\tau - c)\pi i}\right)^j \left(\frac{Q(l)\tau}{\pi i}\right)^u \gamma(u, k-2j)\Theta(A, q_1, l, k-2j, u, \tau)$$

$$= \frac{(-i)^{r+2k}\phi_{h,0}}{c^r\sqrt{D}} \sum_{j=0}^{[k/2]} \sum_{\substack{q_1 \bmod N \\ Aq_1 \equiv 0(N)}} \sum_{u=0}^{[(k-2j)/2]} \sum_{\substack{g \bmod N \\ Ag \equiv 0(N)}} \exp(-2\pi i h' A q_1 b/N^2) \cdot$$

$$\cdot \gamma(j+u, k) \binom{j+u}{j} \left(\frac{c}{d\tau - c}\right)^j \left(\frac{Q(l)\tau}{\pi i}\right)^{j+u} \frac{(-i)^{r+2(k-2j)}}{\sqrt{D}} \cdot$$

$$\cdot \tau^{r+k-2j-2u} \exp(2\pi i g' A q_1/N^2)\theta(A, g, l, k-2j-2u, \tau).$$

Therefore, we get

$$(d\tau - c)^{-(r+k)}\theta(A, h, l, k, \frac{b\tau - a}{d\tau - c})$$

$$= \frac{(-1)^r\phi_{h,0}}{c^r D} \sum_{j=0}^{[k/2]} \sum_{\substack{q_1 \bmod N \\ Aq_1 \equiv 0(N)}} \sum_{u=0}^{[(k-2j)/2]} \sum_{\substack{g \bmod N \\ Ag \equiv 0(N)}} \exp(2\pi i (g - bh)' A q_1/N^2)\gamma(j+u, k) \cdot$$

$$\cdot \binom{j+u}{j} \left(\frac{c}{d\tau - c}\right)^j \left(\frac{Q(l)}{\pi i \tau}\right)^{j+u} \theta(A, g, l, k - 2(j+u), \tau). \tag{3.23}$$

Now we know (page 214 of [S2]) that

$$\sum_{\substack{q_1 \bmod N \\ Aq_1 \equiv 0(N)}} \exp(2\pi i (g - bh)' A q_1/N^2) = D\delta_{g,bh}$$

where $\delta_{g,bh}$ is the Kronecker delta and g, bh are considered modulo N. Thus (3.23) reduces to

$$(d\tau - c)^{-(r+k)}\theta(A, h, l, k, \frac{b\tau - a}{d\tau - c})$$

$$= \frac{(-1)^r\phi_{h,0}}{c^r} \sum_{j=0}^{[k/2]} \sum_{u=0}^{[(k-2j)/2]} \gamma(j+u, k) \binom{j+u}{j} \left(\frac{c}{d\tau - c}\right)^j \left(\frac{Q(l)}{\pi i \tau}\right)^{j+u} \cdot$$

$$\cdot \theta(A, bh, l, k - 2(j+u), \tau)$$

$$= \frac{(-1)^r\phi_{h,0}}{c^r} \sum_{j=0}^{[k/2]} \sum_{t=0}^{[k/2]} \gamma(t, k) \binom{t}{j} \left(\frac{c}{d\tau - c}\right)^j \left(\frac{Q(l)}{\pi i \tau}\right)^t \theta(A, bh, l, k-2t, \tau)$$

$$= \frac{(-1)^r\phi_{h,0}}{c^r} \sum_{t=0}^{[k/2]} \gamma(t, k) \left(\frac{Q(l)}{\pi i \tau}\right)^t \left(\frac{d\tau}{d\tau - c}\right)^t \theta(A, bh, l, k-2t, \tau).$$

Making the change of variables $\begin{pmatrix} b & -a \\ d & -c \end{pmatrix} \mapsto \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ yields the equality

$$(c\tau + d)^{-(r+k)}\theta(A, h, l, k, \frac{a\tau + b}{c\tau + d})$$

$$= \frac{\phi_{h,0}}{d^r} \sum_{t=0}^{[k/2]} \gamma(t,k) \left(\frac{Q(l)c}{\pi i(c\tau+d)} \right)^t \theta(A, bh, l, k-2t, \tau). \tag{3.24}$$

Finally, it is known (page 215 et seq of [S2]) that

$$\frac{\phi_{h,o}}{d^r} = \exp(2\pi i Q(h)ab/N^2)\epsilon(d).$$

Now (3.24) implies all assertions of Theorem 3.4.

We can now complete the proof of Theorem 1. Noting Remark 3.2 (ii) and using (3.16) with $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, we get

$$\Theta(Q, v, \gamma\tau, \frac{X}{(c\tau+d)^2}) = \sum_{n\geq 0} \frac{2^n \theta(Q, v, 2n, \gamma\tau)}{(2n)!} \left(\frac{2\pi i X}{(c\tau+d)^2} \right)^n$$

$$= \epsilon(d)(c\tau+d)^r \sum_{n\geq 0} \sum_{j=0}^{n} \frac{2^n}{(2n)!} \left(\frac{\langle v,v\rangle c}{2\pi i(c\tau+d)} \right)^j \gamma(j,2n)(c\tau+d)^{2n} \cdot$$

$$\cdot \theta(Q, v, 2n-2j, \tau) \left(\frac{2\pi i X}{(c\tau+d)^2} \right)^n$$

$$= \epsilon(d)(c\tau+d)^r \sum_{n\geq 0} \sum_{j=0}^{n} \frac{2^{n-j}}{(2n-2j)!} \theta(Q, v, 2n-2j, \tau)(2\pi i X)^{n-j} \left(\frac{\langle v,v\rangle c}{c\tau+d} \right)^j \frac{X^j}{j!}$$

$$= \epsilon(d)(c\tau+d)^r \exp\left(\frac{\langle v,v\rangle cX}{c\tau+d} \right) \Theta(Q, v, \tau, X),$$

which is the required (1.4) in the case $d > 0$. The general case follows easily.

Finally, we consider behavior at the cusps. This will lead to the proof of Theorem 3 as well as the proof of Theorem 2 initiated in Section 2.

To check the expansion of $\Psi(Q, l, k, \tau)$ at the finite cups, we use Lemma 3.5. Thus for $c > 0$,

$$(c\tau+d)^{-(r+k)} \Psi(Q, l, k, \frac{a\tau+b}{c\tau+d})$$

$$= \sum_{t=0}^{[k/2]} \gamma(t,k)(c\tau+d)^{-2t} E_2(\frac{a\tau+b}{c\tau+d})^t (c\tau+d)^{-(r+k-2t)} \theta(A, l, k-2t, \frac{a\tau+b}{c\tau+d})$$

$$= \sum_{t=0}^{[k/2]} \gamma(t,k) \left(E_2(\tau) - \frac{c}{2\pi i(c\tau+d)} \right)^t \frac{(-i)^{r+2(k-2t)}}{c^r\sqrt{D}} \sum_{j=0}^{[(k-2t)/2]} \sum_{\substack{q \bmod N \\ Aq\equiv 0(N)}} \phi_{0,q} \cdot$$

$$\cdot \gamma(j, k-2t) \left(\frac{cQ(l)}{\pi i(c\tau+d)} \right)^j \theta(A, q, l, k-2t-2j, \tau)$$

$$= \frac{(-i)^{r+2k}}{c^r\sqrt{D}} \sum_{t=0}^{[k/2]} \sum_{j=0}^{[(k-2t)/2]} \sum_{\substack{q \bmod N \\ Aq\equiv 0(N)}} \phi_{0,q}\gamma(j+t,k)\binom{j+t}{j}\left(\frac{c}{2\pi i(c\tau+d)}\right)^j \cdot$$

$$\cdot(2Q(l))^j\left(E_2(\tau)-\frac{c}{2\pi i(c\tau+d)}\right)^t\theta(A,q,l,k-2t-2j,\tau)$$

$$= \frac{(-i)^{r+2k}}{c^r\sqrt{D}} \sum_{\substack{q \bmod N \\ Aq\equiv 0(N)}} \phi_{0,q} \sum_{s=0}^{[k/2]} \gamma(s,k)E_2(\tau)^s\theta(A,q,l,k-2s,\tau). \tag{3.25}$$

This shows that $\Psi(Q,l,k,\tau)$ is holomorphic at the cusps, and thus complete the proof of Theorem 2.

As for Theorem 3, the value of $\Psi(Q,v,2k,\tau)$ at $i\infty$ is the same as that of the function

$$\gamma(k,2k)E_2(\tau)^k\theta(A,v,0,\tau) = \gamma(k,2k)E_2(\tau)^k\theta(Q,\tau)$$

which has q-expansion $\gamma(k,2k)(-1/12)^k(1+\cdots)$. Thus

$$\Psi(Q,v,2k,\tau) - \gamma(k,2k)(-1/12)^k\theta(Q,\tau)E_{2k}(\tau)$$

certainly vanishes at $i\infty$. From (3.25), the value at a general cusp is

$$\frac{(-i)^r}{c^r\sqrt{D}} \sum_{\substack{q \bmod N \\ Aq\equiv 0(N)}} \phi_{0,q}\gamma(k,2k)(-1/12)^k.$$

But $\frac{(-i)^r}{c^r\sqrt{D}} \sum_{\substack{q \bmod N \\ Aq\equiv 0(N)}} \phi_{0,q}$ is the value of $\theta(Q,\tau)$ at the same cusp (cf. equation (24) of [S2], page 213), whence it is clear that $\Psi(Q,v,2k,\tau) - \gamma(k,2k)(-1/12)^k\theta(Q,\tau)E_{2k}(\tau)$ vanishes at every cusp if $k \geq 2$. This completes the proof of Theorem 3.

4 Concluding comments

We have already mentioned that the previous results were motivated by conformal field theory, more precisely by the problem of calculating 1-point correlation functions for vertex operator algebras [DMN]. For earlier results in this direction, see [DLM] and [DM]. This perspective also enables us to prove the following result (see [DMN]): let the notation be as before, and suppose that the lattice $\mathbb{Z}^f$ contains a root α ie., $Q(\alpha)=1$. Then the cusp form of Theorem 3 (with $v=\frac{\alpha}{\sqrt{2}}$) is identically zero. That is, we have

$$\theta(Q,\frac{\alpha}{\sqrt{2}},4,\tau) + 6E_2(\tau)\theta(Q,\frac{\alpha}{\sqrt{2}},2,\tau) + 3E_2(\tau)^2\theta(Q,\tau) = \frac{1}{48}E_4(\tau)\theta(Q,\tau). \tag{4.1}$$

It is interesting that in [Z], Zagier raises the question of whether there is a relation between Jacobi-like forms and vertex operator algebras. The present work together with [DMN] certainly suggests that this question continues to be one which is worth exploring.

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